

Property A and the existence of a Markov process with a trivial Poisson boundary

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Abstract. This note make the observation that property A for a space is equivalent to the existence of a Markov process on the space which has a (uniformly) trivial Poisson boundary.

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1 Introduction

In [Yu00], Yu introduced property A which can be viewed as a non-equivariant analogue of amenability. To explain what "non-equivariant" means, we recall Reiter's necessary and sufficient condition for amenability:

Theorem 1.1. *A countable group G is amenable iff there exists a sequence of non degenerate probability measures μ_n on G such that*

$$\forall g \in G, \lim_{n \rightarrow \infty} \|g\mu_n - \mu_n\|_1 = 0$$

One can think of the sequence of measures in the above theorem as a sequence of maps $\phi_n : G \rightarrow \text{Prob}(G)$ that is defined as $\phi_n(g, h) = \mu_n(g^{-1}h)$, $\forall g \in G$ (we denote $\phi_n(g)(h) = \phi(g, h)$). Compare this condition to the condition of Higman and Roe [HR00] for property A:

Theorem 1.2. *A finitely generated group has property A iff there exists a sequence of maps $\phi_n : G \rightarrow \text{Prob}(G)$ with the following properties:*

1. *For every n there is a finite subset $F \subset G$ such that for every $g \in G$ we have $g^{-1}\text{Supp}(\phi_n(g, \cdot)) \subset F$.*

2. For every $g \in G$ we have

$$\lim_{n \rightarrow \infty} \sup_{h \in G} \|\phi_n(h, \cdot) - \phi_n(hg, \cdot)\|_1 = 0$$

One can see the analogy between the two theorems that is not full due to the fact that in the condition for amenability, μ_n doesn't have to be of finite support, but this can easily be fixed as we shall see later, by replacing the finite support condition of the map ϕ_n in theorem 1.2 by the condition:

For every n and for every $\delta > 0$ there is a finite set $F_\delta \subset G$ such that for every $g \in G$ we have

$$\sum_{h \in F_\delta} \phi_n(g, h) > 1 - \delta$$

In response to a conjuncture by Furstenberg, it was proven in [Ros81] and in [KV83] that the sequence of measures in the condition for amenability can be given as a sequence of convolutions of a single measure:

Theorem 1.3. *A countable group G is amenable iff there exist a probability measure μ on G such that*

$$\forall g \in G, \lim_{n \rightarrow \infty} \|g\mu^{*n} - \mu^{*n}\|_1 = 0$$

it is also shown at [Ros81], [KV83] this condition for amenability is equivalent to the triviality of the Poisson boundary of (G, μ) .

The easy observation made in this note is that the technique used in [KV83] needs very little adaptation to the case of property A, namely we show that:

Theorem 1.4. *A finitely generated group has property A iff there exists a transition probability P with the state space G with the following properties:*

1. For every $\delta > 0$ there is a finite set $F_\delta \subset G$ such that for every $g \in G$ we have

$$\sum_{h \in F_\delta} P(g, gh) > 1 - \delta$$

2. For every $g \in G$ we have

$$\lim_{n \rightarrow \infty} \sup_{h \in G} \|P^n(h, \cdot) - P^n(hg, \cdot)\|_1 = 0$$

The second condition above implies the triviality of the Poisson boundary of the Markov chain defined by P and any initial probability of X , but it is not equivalent to it. One can think of this condition as a metric uniform version of the condition for triviality of the Poisson boundary (the sufficient condition for triviality of the Poisson boundary of the Markov chain for any initial probability, requires a convergence of the sequence of products that is much more tame).

The above theorem is stated for finitely generated groups, but applies in a more general metric setting of bounded geometry. The structure of this note is as following - in section 2 we will give the necessary background on bounded geometry, property A and the Poisson boundary of a Markov process (this background will be far from complete, since a complete background on any of the mentioned topics is far beyond the scope of this note). In section 3 we will prove theorem 1.4 for the general case of metric spaces with bounded geometry.

2 Background

2.1 Markov chains and the Poisson boundary

2.1.1 Measures and convolutions

Let X be a countable set, and let $Prob(X)$ be the space of functions $\mu : X \rightarrow \mathbb{R}_{\geq 0}$ such that $\sum_x \mu(x) = 1$. The L^1 norm on $Prob(X)$ is defined as usual to be

$$\|\mu - \nu\|_1 = \sum_x |\mu(x) - \nu(x)|$$

(since we are dealing with a countable space, this norm is equivalent to the total variation norm).

A transition probability on the state space X is a map $\phi : X \rightarrow Prob(X)$ so $\phi(x) \in Prob(X)$ and we shall use the notation $\phi(x)(y) = \phi(x, y)$.

Given a measure $\mu \in Prob(X)$ and a transition probability ϕ on X , the convolution $\mu * \phi \in Prob(X)$ is defined as

$$(\mu * \phi)(y) = \sum_x \mu(x) \phi(x, y)$$

Given two transition probabilities ϕ, ψ on X , the convolution $\phi * \psi$ is a transition probability on X defined as

$$(\phi * \psi)(x, y) = \sum_z \phi(x, z) \psi(z, y)$$

For a transition probability P we shall use the notation $P^n = P * \dots * P$ and $P^n(x, \cdot)$ is $\delta_x * P^n$. Below is an easy proposition about the interplay between convolution and the L^1 norm (the proof is given for the sake of completeness).

Proposition 2.1. *Let $\mu, \nu \in Prob(X)$ and ϕ a transition probability on X , then*

1. *Convolution (from the right) with ϕ is a non expanding map, i.e.*

$$\|\mu * \phi - \nu * \phi\|_1 \leq \|\mu - \nu\|_1$$

2. *If there is a $\varepsilon > 0$ and an $x_0 \in X$ such that for all $x \in \text{Supp}(\mu) \cup \text{Supp}(\nu)$ we have $\|\phi(x, \cdot) - \phi(x_0, \cdot)\|_1 < \varepsilon$ then*

$$\|\mu * \phi - \nu * \phi\|_1 < 2\varepsilon$$

Proof. 1. For every $\mu, \nu \in Prob(X)$ we have

$$\begin{aligned} \|\mu * \phi - \nu * \phi\|_1 &= \sum_y \left| \sum_x \mu(x) \phi(x, y) - \sum_x \nu(x) \phi(x, y) \right| \leq \\ &\leq \sum_y \sum_x |(\mu(x) - \nu(x)) \phi(x, y)| = \sum_x |\mu(x) - \nu(x)| = \|\mu - \nu\|_1 \end{aligned}$$

2. Notice that since $\mu, \nu \in \text{Prob}(X)$ we have for any y that

$$\phi(x_0, y) = \sum_x \mu(x) \phi(x_0, y) = \sum_x \nu(x) \phi(x_0, y)$$

So

$$\begin{aligned} \|\mu * \phi - \nu * \phi\|_1 &= \sum_y \left| \sum_x \mu(x) \phi(x, y) - \sum_x \nu(x) \phi(x, y) \right| = \\ &= \sum_y \left| \sum_x \mu(x) \phi(x, y) - \sum_x \mu(x) \phi(x_0, y) + \sum_x \nu(x) \phi(x_0, y) - \sum_x \nu(x) \phi(x, y) \right| \leq \\ &\leq \sum_y \left| \sum_x \mu(x) (\phi(x, y) - \phi(x_0, y)) \right| + \sum_y \left| \sum_x \nu(x) (\phi(x, y) - \phi(x_0, y)) \right| \leq \\ &\leq \sum_{x \in \text{Supp}(\mu)} \mu(x) \sum_y |\phi(x, y) - \phi(x_0, y)| + \sum_{x \in \text{Supp}(\nu)} \nu(x) \sum_y |\phi(x, y) - \phi(x_0, y)| \leq \\ &< \sum_{x \in \text{Supp}(\mu)} \mu(x) \varepsilon + \sum_{x \in \text{Supp}(\nu)} \nu(x) \varepsilon = 2\varepsilon \end{aligned}$$

□

2.1.2 The Poisson boundary of a Markov chain

A triple (X, μ, P) where X is a countable state space, $\mu \in \text{Prob}(X)$ and P is a transition probability on X is called as a (time homogeneous) Markov chain. For a Markov chain (X, μ, P) , μ is called the initial probability of the chain. Given a countable state space X , define the measure space $X^{\mathbb{N}}$ (the sequences of elements of X) with the σ -algebra \mathcal{A} generated by the cylinder sets:

$$[x_1, \dots, x_n] = \{\omega = (y_1, \dots) \in X^{\mathbb{N}} : y_1 = x_1, \dots, y_n = x_n\}$$

A Markov chain (X, μ, P) defines a probability measure P_μ on $X^{\mathbb{N}}, \mathcal{A}$ given as

$$P_\mu([x_1, \dots, x_n]) = \mu(x_1)P(x_1, x_2) \dots P(x_{n-1}, x_n)$$

Define the operator $T : X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$ as

$$T(x_1, x_2, \dots) = (x_2, x_3, \dots)$$

Then the Poisson boundary of (X, μ, P) is defined as the ergodic components of T under the measure P_μ . We say that the Poisson boundary is trivial if there is only one ergodic component. In [Der76], Derriennic proved the following (see also [Kai92]):

Theorem 2.2. *Given X and a transition probability P , the Poisson boundary of (X, μ, P) is trivial for all $\mu \in \text{Prob}(X)$ iff for every $x, y \in X$ we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\| \sum_{i=1}^n P^i(x, \cdot) - \sum_{i=1}^n P^i(y, \cdot) \right\|_1 = 0$$

When $X = G$ is a countable group, the Poisson boundary is defined with respect to a transition probability P which is invariant of the group action, i.e.

$$\forall h, g \in G, P(g, gh) = P(e, h)$$

Rosenblatt [Ros81] and Kaimanovich-Vershik [KV83] proved the following theorem characterizing Amenable groups in terms of the Poisson boundary:

Theorem 2.3. *A countable group G is discrete iff there is an invariant transition probability P on G such that the Poisson boundary of (G, μ, P) is trivial for all $\mu \in \text{Prob}(G)$.*

2.2 Bounded geometry and Property A

In this section will give the basic definitions regarding metric spaces with bounded geometry and Property A. Throughout this entire paper we assume that our metric space (X, d) is discrete and countable.

Definition 2.4. *A discrete metric space (X, d) is said to have bounded geometry if for every $C > 0$ there is a number $M(C)$ such that for every $x \in X$ we have*

$$|B(x, C)| \leq M(C)$$

The following example is was of the main motivations to study discrete metric spaces with bounded geometry.

Example 2.5. *Let G be a finitely generated group with a generating set S , then it is obvious the Cayley graph of G with respect to S with its usual metric is a discrete metric space with bounded geometry.*

Definition 2.6. *A discrete metric space (X, d) is said to have Property A if there is a collection $\{A_x^n\}_{x \in X, n \in \mathbb{N}}$ of finite subsets of $X \times \mathbb{N}$ such that the following holds:*

1. *For every $n \in \mathbb{N}$ there is a number R_n such that for every $x \in X$ we have*

$$A_x^n \subset B(x, R_n) \times \mathbb{N}$$

2. *For every $K > 0$ we have that*

$$\lim_{n \rightarrow \infty} \sup_{d(x, y) < K} \frac{|A_x^n \triangle A_y^n|}{|A_x^n \cap A_y^n|} = 0$$

In [HR00] Higman and Roe gave a characterization of property A that is analogues to Reiter's condition for amenability. Namely they proved the following (when this theorem is applied to the example of finitely generated groups, one gets theorem 1.2):

Theorem 2.7. *Let (X, d) be a countable discrete metric space with bounded geometry, then (X, d) has property A iff there is a sequence of transition probabilities $\phi_n : X \rightarrow \text{Prob}(X)$ such that the following holds:*

1. *For every n there is a number R_n such that for every $x \in X$ we have*

$$\text{Supp}(\phi_n(x, \cdot)) \subset B(x, R_n)$$

2. For every $K > 0$ we have

$$\lim_{n \rightarrow \infty} \sup_{d(x,y) < K} \|\phi_n(x, \cdot) - \phi_n(y, \cdot)\|_1 = 0$$

In the next proposition, we show that the first condition in the above theorem can be weakened:

Proposition 2.8. *Let (X, d) be a countable discrete metric space with bounded geometry, then (X, d) has property A iff there is a sequence of transition probabilities $\phi_n : X \rightarrow \text{Prob}(X)$ such that the following holds:*

1. For every n and for every $\delta > 0$ there is there is a number $R_{\delta, n}$ such that for every $x \in X$ we have

$$\sum_{y \in B(x, R_{\delta, n})} \phi_n(x, y) > 1 - \delta$$

2. For every $K > 0$ we have

$$\lim_{n \rightarrow \infty} \sup_{d(x,y) < K} \|\phi_n(x, \cdot) - \phi_n(y, \cdot)\|_1 = 0$$

Proof. The first direction is obvious from theorem 2.7 - for every n and every $\delta > 0$ choose $R_{\delta, n} = R_n$. Conversely, assume there is a sequence of transition probabilities $\phi_n : X \rightarrow \text{Prob}(X)$ with the conditions stated above and define a new sequence ϕ'_n as follows:

$$\phi'_n(x, y) = \begin{cases} \frac{1}{\sum_{y \in B(x, R_{\frac{1}{n}, n})} \phi_n(x, y)} \phi_n(x, y) & y \in B(x, R_{\frac{1}{n}, n}) \\ 0 & d(x, y) \geq R_{\frac{1}{n}, n} \end{cases}$$

Then ϕ'_n is a sequence of transition probability such that for every $x \in X$ we have

$$\text{Supp}(\phi'_n(x, \cdot)) \subset B(x, R_{\frac{1}{n}, n})$$

and also for every $n > 1$ and every $x \in X$ we have that

$$\begin{aligned} \|\phi_n(x, \cdot) - \phi'_n(x, \cdot)\|_1 &= \sum_{y \in B(x, R_{\frac{1}{n}, n})} \phi_n(x, y) \left(\frac{1}{\sum_{y \in B(x, R_{\frac{1}{n}, n})} \phi_n(x, y)} - 1 \right) + \\ &+ \sum_{y, d(x, y) \geq R_{\frac{1}{n}, n}} \phi_n(x, y) \leq \frac{1}{1 - \frac{1}{n}} - 1 + \frac{1}{n} \leq \frac{3}{n} \end{aligned}$$

So for every $x, y \in X$ and every $n > 1$ we have that

$$\|\phi'_n(x, \cdot) - \phi'_n(y, \cdot)\|_1 \leq \|\phi'_n(x, \cdot) - \phi_n(x, \cdot)\|_1 +$$

$$+\|\phi_n(x, \cdot) - \phi_n(y, \cdot)\|_1 + \|\phi_n(y, \cdot) - \phi'_n(y, \cdot)\|_1 \leq \|\phi_n(x, \cdot) - \phi_n(y, \cdot)\|_1 + \frac{6}{n}$$

and therefore

$$\lim_{n \rightarrow \infty} \sup_{d(x,y) < K} \|\phi'_n(x, \cdot) - \phi'_n(y, \cdot)\|_1 \leq \lim_{n \rightarrow \infty} \left(\sup_{d(x,y) < K} \|\phi_n(x, \cdot) - \phi_n(y, \cdot)\|_1 + \frac{6}{n} \right) = 0$$

and the conditions of theorem 2.7 hold for ϕ'_n . \square

3 Property A as uniform triviality of the Poisson boundary

Theorem 3.1. *Let (X, d) be a countable, discrete metric space with a bounded geometry. Then (X, d) has property A iff there exists a transition probability P on the state space X with the following properties:*

1. For every $\delta > 0$ there is some R_δ such that for every $x \in X$ we have

$$\sum_{y \in B(x, R_\delta)} P(x, y) > 1 - \delta$$

2. For every $K > 0$ we have that

$$\lim_{n \rightarrow \infty} \sup_{d(x, y) < K} \|P^n(x, \cdot) - P^n(y, \cdot)\|_1 = 0$$

Proof. Let (X, d) be a countable, discrete metric space with a bounded geometry and assume that (X, d) has property A, so by theorem 2.7 there is a sequence of maps $\phi_n : X \rightarrow \text{Prob}(X)$ such that:

1. For every n there is R_n such that for all $x \in X$ we have that $\text{Supp}(\phi_n(x, \cdot)) \subset B(x, R_n)$.
2. For every $K > 0$ we have that

$$\lim_{n \rightarrow \infty} \sup_{d(x, y) < K} \|\phi_n(x, \cdot) - \phi_n(y, \cdot)\|_1 = 0$$

Choose two sequences of positive real numbers $\{t_i\}, \{\varepsilon_i\}$ such that $\sum_i t_i = 1$ and $\lim_{i \rightarrow \infty} \varepsilon_i = 0$. Let $\{n_i\}$ be an increasing sequence of natural numbers such that

$$(t_1 + \dots + t_{i-1})^{n_i} < \varepsilon_i$$

Define a subsequence of ϕ_i which we will also denote ϕ_i inductively: let ϕ^1 be the map such that for every $x, y \in X$ with $d(x, y) < 1$ we have

$$\|\phi_1(x, \cdot) - \phi_1(y, \cdot)\|_1 < \varepsilon_1$$

and let R_1 be a number such that $\text{Supp}(\phi_1(x, \cdot)) \subset B(x, R_1)$ for every $x \in X$. Define ϕ_i to be the transition probability such that for every $x, y \in X$ with $d(x, y) < n_i R_{i-1}$ we have

$$\|\phi_i(x, \cdot) - \phi_i(y, \cdot)\|_1 < \varepsilon_i$$

and let R_i be a number such that $\text{Supp}(\phi_i(x, \cdot)) \subset B(x, R_i)$ for every $x \in X$ and WLOG we choose $R_i > \max\{R_{i-1}, i\}$.

We shall show that $P = \sum_i t_i \phi_i$ is a transition probability with the properties stated in the theorem. First note that for every $\delta > 0$ there is some i_0 such that $\sum_{i=i_0}^\infty t_i < \delta$. Choose $R_\delta = R_{i_0}$ - for every $x \in X$ we have that $\text{Supp}(\phi_i(x, \cdot)) \subset B(x, R_i)$, and since we choose R_i to be monotone increasing, we have

$$\forall i < i_0, \forall x, y \in X, d(x, y) \geq R_{i_0} \Rightarrow \phi_i(x, y) = 0$$

Therefore

$$\sum_{y, d(x, y) \geq R_{i_0}} P(x, y) = \sum_{y, d(x, y) \geq R_{i_0}} \sum_{i=i_0}^{\infty} t_i \phi_i(x, y) \leq \sum_{i=i_0}^{\infty} t_i < \delta$$

So we got that

$$\sum_{y \in B(x, R_{i_0})} P(x, y) > 1 - \delta$$

To prove the second condition we shall show that for every $x, y \in X$ with $d(x, y) < R_{i-1}$ we have

$$\|P^{n_i}(x, \cdot) - P^{n_i}(y, \cdot)\|_1 \leq 4\varepsilon_i$$

(it is sufficient to work with the subsequence n_i because from 2.1 we get that for every $n > n_i$ we have

$$\|P^n(x, \cdot) - P^n(y, \cdot)\|_1 \leq \|P^{n_i}(x, \cdot) - P^{n_i}(y, \cdot)\|_1$$

).

By definition we have that

$$P^{n_i}(\cdot, \cdot) = \sum_{(k_1, \dots, k_{n_i}) \in \mathbb{N}^{n_i}} t_{k_1} \dots t_{k_{n_i}} \phi_{k_1} * \dots * \phi_{k_{n_i}}$$

define

$$A_i = \{(k_1, \dots, k_{n_i}) \in \mathbb{N}^{n_i} : k_j < i, \forall j\}$$

$$B_i = \mathbb{N}^{n_i} \setminus A_i$$

So

$$P^{n_i}(\cdot, \cdot) = \sum_{(k_1, \dots, k_{n_i}) \in A_i} t_{k_1} \dots t_{k_{n_i}} \phi_{k_1} * \dots * \phi_{k_{n_i}} + \sum_{(k_1, \dots, k_{n_i}) \in B_i} t_{k_1} \dots t_{k_{n_i}} \phi_{k_1} * \dots * \phi_{k_{n_i}}$$

Note that for every $x \in X$

$$\begin{aligned} & \left\| \sum_{(k_1, \dots, k_{n_i}) \in A_i} t_{k_1} \dots t_{k_{n_i}} \phi_{k_1} * \dots * \phi_{k_{n_i}}(x, \cdot) \right\|_1 \leq \\ & \leq \sum_{(k_1, \dots, k_{n_i}) \in A_i} t_{k_1} \dots t_{k_{n_i}} = (t_1 + \dots + t_{i-1})^{n_i} < \varepsilon_i \end{aligned}$$

It follows that for every $x, y \in X$ we have

$$\begin{aligned} & \|P(x, \cdot) - P(y, \cdot)\|_1 < 2\varepsilon_i + \\ & + \left\| \sum_{(k_1, \dots, k_{n_i}) \in B_i} t_{k_1} \dots t_{k_{n_i}} ((\phi_{k_1} * \dots * \phi_{k_{n_i}})(x, \cdot) - (\phi_{k_1} * \dots * \phi_{k_{n_i}})(y, \cdot)) \right\|_1 \leq \\ & \leq 2\varepsilon_i + \sum_{(k_1, \dots, k_{n_i}) \in B_i} t_{k_1} \dots t_{k_{n_i}} \|(\phi_{k_1} * \dots * \phi_{k_{n_i}})(x, \cdot) - (\phi_{k_1} * \dots * \phi_{k_{n_i}})(y, \cdot)\|_1 \end{aligned}$$

So to show

$$\|P(x, \cdot) - P(y, \cdot)\|_1 < 4\varepsilon_i$$

it is sufficient to show that for every $(k_1, \dots, k_{n_i}) \in B_i$ we have

$$\|(\phi_{k_1} * \dots * \phi_{k_{n_i}})(x, \cdot) - (\phi_{k_1} * \dots * \phi_{k_{n_i}})(y, \cdot)\|_1 \leq 2\varepsilon_i$$

Let $(k_1, \dots, k_{n_i}) \in B_i$ and let j be the largest index such that $k_j < i$ (so $1 \leq j < n_i$). Denote

$$\mu = \phi_{k_1} * \dots * \phi_{k_j}(x, \cdot) \in \text{Prob}(X)$$

$$\nu = \phi_{k_1} * \dots * \phi_{k_j}(y, \cdot) \in \text{Prob}(X)$$

Then we need to show

$$\|\mu * \phi_{k_{j+1}} * \dots * \phi_{k_{n_i}} - \nu * \phi_{k_{j+1}} * \dots * \phi_{k_{n_i}}\|_1 < 2\varepsilon_i$$

and by proposition 2.1 it is enough to show

$$\|\mu * \phi_{k_{j+1}} - \nu * \phi_{k_{j+1}}\|_1 < 2\varepsilon_i$$

Since $k_1, \dots, k_j < i$ we have that $\text{Supp}(\mu) \subset B(x, jR_{i-1})$ and $\text{Supp}(\nu) \subset B(y, jR_{i-1})$. Also, since $d(x, y) < R_{i-1}$ we have that $\text{Supp}(\nu) \subset B(x, (j+1)R_{i-1}) \subset B(x, n_i R_{i-1})$ and obviously $\text{Supp}(\mu) \subset B(x, n_i R_{i-1})$. Since $k_{j+1} \geq i$ we have from the choice of $R_{k_{j+1}} \geq R_i \geq n_i R_{i-1}$ that for all $x' \in B(x, n_i R_{i-1})$ we have

$$\|\phi_{k_{j+1}}(x, \cdot) - \phi_{k_{j+1}}(x', \cdot)\|_1 < \varepsilon_i$$

Applying proposition 2.1 we get that

$$\|\mu * \phi_{k_{j+1}} - \nu * \phi_{k_{j+1}}\|_1 < 2\varepsilon_i$$

and the proof of the first direction is complete.

The other direction follows from proposition 2.8 - if there is a transition probability P with the above properties, define $\phi_n : X \rightarrow \text{Prob}(X)$ as $\phi_n(x, \cdot) = P^n(x, \cdot)$. We need only to show that for every n and every $\delta > 0$ there is some $R_{\delta, n}$ such that for every $x \in X$ we have

$$\sum_{y \in B(x, R_{\delta, n})} P^n(x, y) > 1 - \delta$$

If $n = 1$ by the conditions of the theorem for every $\delta > 0$ we have $R_{\delta, 1}$ that meets the above requirement. Fix $\delta > 0$, $n > 1$ and choose $R_{\delta, n} = nR_{\frac{\delta}{n}, 1}$. Indeed observed that for every $x \in X$ we have

$$\begin{aligned} & \sum_{y, d(x, y) \geq R_{\delta, n}} P^n(x, y) \leq \\ & \leq \sum_{i=1}^n \sum_{z_1 \in X} P^{i-1}(x, z_1) \sum_{z_2 \in X, d(z_1, z_2) \geq R_{\frac{\delta}{n}, 1}} P(z_1, z_2) \sum_{y \in X} P^{n-i}(z_2, y) \leq \delta \end{aligned}$$

□

When the theorem above is applied to the case of finitely generated groups one get theorem 1.4 stated in the introduction.

Remark 3.2. The sufficient condition for the triviality of the Poisson boundary of X, P of theorem 2.2 can be read in the following way: the Poisson boundary of X, P is trivial for any initial probability iff the sequence of transition probabilities defined as

$$\phi_n(x, \cdot) = \frac{1}{n} (P(x, \cdot) + \dots + P^n(x, \cdot))$$

satisfies the condition

$$\forall x, y \in X, \lim_{n \rightarrow \infty} \|\phi_n(x, \cdot) - \phi_n(y, \cdot)\|_1 = 0$$

The above condition doesn't require X to be a metric space and therefore doesn't take into account any interplay between metric and measure. Comparing the above condition to the conditions given in theorem 3.1, property A is equivalent to stronger conditions:

1. The needed limit is of convolutions and not an average of convolutions (see [Ros81] to the difference between "ergodic by convolutions" and "mixing by convolutions").
2. The limit has uniformity with respect to the metric.

Therefore one can think of the conditions of theorem 3.1 as "uniform" triviality of the Poisson boundary. One should note, that without the uniformity in the condition, it becomes trivial as stated in the next proposition.

Proposition 3.3. For every metric space (X, d) there is a transition probability P with

1. For every $\delta > 0$ there is some R_δ such that for every $x \in X$ we have

$$\sum_{y \in B(x, R_\delta)} P(x, y) > 1 - \delta$$

2. For every $x, y \in X$ we have that

$$\lim_{n \rightarrow \infty} \|P^n(x, \cdot) - P^n(y, \cdot)\|_1 = 0$$

Proof. Fix some $x_0 \in X$ and define

$$P(x, \cdot) = \begin{cases} \delta_{x_0} & x = x_0 \\ \frac{1}{1+d(x, x_0)} \delta_{x_0} + \frac{d(x, x_0)}{1+d(x, x_0)} \delta_x & x \neq x_0 \end{cases}$$

Then for every $\delta > 0$ take $R_\delta = \frac{1}{\delta}$ and check that

$$\sum_{y \in B(x, R_\delta)} P(x, y) > 1 - \delta$$

Also, for every $x \neq x_0$ we have

$$P^n(x, \cdot) = (1 - (\frac{d(x, x_0)}{1+d(x, x_0)})^n) \delta_{x_0} + (\frac{d(x, x_0)}{1+d(x, x_0)})^n \delta_x$$

and therefore we have for every $x, y \in X$ that

$$\|P^n(x, \cdot) - P^n(y, \cdot)\|_1 \leq 2(\frac{d(x, x_0)}{1+d(x, x_0)})^n + 2(\frac{d(y, x_0)}{1+d(y, x_0)})^n$$

and the proposition is proved. \square

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